Geometry Problems from the IMO Shortlist

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1 Introduction

These notes outline a few basic synthetic methods to approach geometry problems on Olympiads. The solution outlines are sketches intended to emphasize motivation more than rigour. In particular, they do not deal with configuration issues and special cases. The accompanying handout includes a few geometry facts and theorems that are useful to know and also worthwhile exercises to work through an prove yourself. Some of the more useful of these facts are highlighted in Section 6.

2 Ideas to Try

Ideas to try on geometry problems:

1. *Angle Chasing*: Choose a set of angles that defines the diagram and find all possible angles in terms of them *e.g.* using cyclic quadrilaterals, similar triangles, common angle formulas.

2. *Length Chasing*: Find relationships between the lengths of sides *e.g.* using power of a point, similar triangles, Menelaus and Ceva, incircle and excircle side lengths, Pythagorean Theorem.

3. *Reduce the Problem*: Make some observations that reduce the problem to an easier problem or conjecture something plausible that implies the problem statement.

4. *Phantom Points*: To prove that a point P has a property, define a new point P' in a way that is easier to work with, then prove the property for P' and prove that $P = P'$.

5. *Combine Patterns*: Bring parts of the diagram that are related to each other together *e.g.* through parallel lines, intersecting circumcircles, reflections, constructing similar triangles.

6. *Spiral Similarity*: Look for or construct similar triangles of the form AOB and COD and use the angle and length relationships from the fact that AOC and BOD are also similar.

7. *Transformations*: Look for any transformations already present in the diagram and apply them to other parts of the diagram *e.g.* homothety, translation, reflection, spiral similarity.

8. *Constructing Points*: An introduced point P generally is useful if it has two “good” properties *i.e.* unites two conditions in the problem. Since a point P can always be selected to have a single property, introducing a point is only useful when it unites two conditions. However, most points introduced to solve problems are likely motivated by an approach listed above.

9. *Forming Conjectures*: Many difficult problems will require a lemma which may not be obvious from the problem statement or initial deductions. Two ways of forming conjectures are:
   (a) looking for patterns in precisely drawn diagrams and
   (b) thinking about what would be convenient and easy to work with if it were true

   It is important to keep both of these ideas in mind when looking for a key observation. Observations made only from the diagram may not be feasible to prove, useless to the problem or false. Conjectures that would be convenient may be obviously disproved by a diagram. It is important to conjecture something which seems clearly true based on one good (or several) diagrams and is both feasible to prove and useful in the problem.

10. *What is Difficult?*: A diagram will likely have parts that are difficult and parts that are easier to work with. It is often useful to identify what parts are difficult to work with and try to figure out possible ways to handle them *e.g.* redefining points using phantom points.

11. *Trigonometry*: Powerful in situations when an angle cannot be expressed simply in terms of other angles *e.g.* angles involving medians; often works best when you have in mind exactly what you want to prove *e.g.* a ratio condition.

12. *Algebraic Methods*: Complex numbers, vectors, coordinates and barycentric coordinates.

3 Exhaust the Diagram

Sometimes it is tempting with geometry problems to immediately start guessing what magic point, line or circle to draw in the diagram leads to an elegant short solution. Before doing this, it is worthwhile to make sure that the problem actually needs something new. Oftentimes, the problem statement already has introduced all parts of the diagram that the most straightforward solution will need. In these cases, making sure you figure out everything you can with what you are given is much more productive than adding points to the diagram. Here are some things to try:

1. *Angle Chasing*: Given your knowledge of similar triangles and cyclic quadrilaterals in the diagram, find all angle relationships you can in the diagram. This is an essential step in almost all Olympiad geometry problems.

2. *Length Chasing*: Many problems can be solved by alternating between angle and length chasing – using some length relationships to find a new cyclic quadrilateral or pair of similar triangles and subsequently making use of whatever new angle relationships this yields. Here are some approaches to length chasing:
   (a) *Similar Triangles*: These arise in many different contexts. One common way is spiral similarity: If OAB and OCD are similar triangles with the same orientation, then so are OAC and OBD.
   (b) *Power of a Point*: If AB and CD meet at the point P then $PA \cdot PB = PC \cdot PD$ if and only if $ABCD$ is cyclic.

(c) *Menelaus and Ceva*: Given a triangle ABC, let the points D, E and F be on lines AB, BC and AC, respectively. Then

$$\frac{AD}{DB} \cdot \frac{BE}{EC} \cdot \frac{CF}{FA} = 1$$

if and only if D, E and F are collinear or CD, AE and BF are concurrent (depending on how many of D, E and F are on the sides of ABC).

3. *Work Backwards*: Assuming the result is true, what else would have to be true? Can you show any of these implications without assuming the result? Can you use any of these intermediary results to solve the problem?

Here are some examples of problems that can be solved by exhausting the diagram as given in the problem statement, without adding any drastically new points. The first two examples need no new points at all.

Example 1. *(ISL 2008 G4) In an acute triangle ABC segments BE and CF are altitudes. Two circles passing through the points A and F and tangent to the line BC at the points P and Q so that B lies between C and Q. Prove that lines PE and QF intersect on the circumcircle of triangle AEF.*

Solution. This problem is straightforward with power of a point and does not require introducing any new points other than the orthocenter H of ABC and foot of the perpendicular from A to BC, which are already implicitly present. Relating our goal to angles already in the diagram reduces the problem to showing that $\angle QFB = \angle PEC$. By power of a point $BQ^2 = BP^2 = BF \cdot BA$ and triangles QFB and AQB are similar. Therefore it suffices to show that $\angle PEC = \angle AQC$ which is equivalent to $AQPE$ being cyclic. By power of a point we now have

$$CP \cdot CQ = BC^2 - BP^2 = BC^2 - BF \cdot BA = BC^2 - BD \cdot BC = CD \cdot CB = CE \cdot CA$$

Therefore $AQPE$ is cyclic and we are done. □

4 Completing the Diagram

As seen in the last few examples in the previous section, it is often useful to introduce some points implicit in the problem statement, such as intersection points, triangle centers and projections. A large number of Olympiad geometry problems can be solved by (1) exhausting the diagram and (2) in this way "completing the diagram". This is a vague heuristic and can take many forms, which are impossible to characterize in a single broad stroke. Nonetheless, here is an attempt at some intuition as to when completing the diagram can be useful.

1. *The Triangle Picture*: Add in orthocenters, circumcenters, excenters, incenters, the circumcircle, midpoints of arcs, feet of altitudes, etc. depending on whether they clarify any parts of the problem statement. This is almost always a good idea to at least try.

2. *Intersecting Lines*: This can be useful, especially when the intersection is at an angle that can be calculated or has some other significance. A somewhat trivial-sounding rule of thumb is that you want to introduce intersections that add clarity rather than further complicate the diagram. Usually one or several pairs of lines will stand out as useful to intersect.

3. *Intersecting Lines with Circles*: This is often useful since circles generally give angle relationships for free.

4. *Implicit Circles*: Sometimes an angle relationship or length relationship will be best simplified when interpreted in terms of a hidden circle.

5. *Parallel and Perpendicular Lines*: Sometimes it is useful to project points onto lines, either with a perpendicular or skew projection with parallel lines. This is often to create similar triangles or measure lengths.

Another somewhat trivial-sounding rule of thumb is that a introducing a line, point or circle to a diagram is only useful if it was implicit in the problem statement or **relates two objects that were previously not relatable**. This is the entire heuristic motivation behind the "completing the transformation" tricks for finding new points that are in the next section. In the sections afterwards, we discuss more heuristics in finding the right points to add to a diagram, including phantom points and intersecting circles. This section is devoted to more generic ways to add points to a diagram, which we demonstrate through several miscellaneous examples.

Example 2. *(ISL 2006 G4) Let ABC be a triangle such that $\widehat{ACB} < \widehat{BAC} < \frac{\pi}{2}$. Let D be a point of $[AC]$ such that $BD = BA$. The incircle of ABC touches $[AB]$ at K and $[AC]$ at L. Let J be the center of the incircle of BCD. Prove that (KL) intersects $[AJ]$ at its middle.*

Solution. Angle chasing gives that $\angle ALK = 90^\circ - \angle A/2$ and $\angle CDJ = 90^\circ - \angle A/2$. It makes sense to try to relate these two equal angles in the diagram by trying to move one into a position so that it relates to the other. Furthermore, working on the segment AJ seems difficult as we do not know angles or lengths related to this line. Instead, we try to work on AC, where we can make use of incircle tangent length formulas. We do this by reducing the problem using non-perpendicular projections in the direction of KL onto AC. We find that this reduces the problem to a seemingly feasible alternative and also relates the equal angles originally found. Specifically, let P be the intersection of the line perpendicular to KL through J with AC. It now suffices to show that L is the midpoint of AP. Since $\angle PDJ = \angle ALK = \angle DPJ$, we have that PDJ is isosceles and if M is the midpoint of DP, then M is also the foot of the perpendicular from J onto AC. Applying incircle tangent length formulas gives that $AL = \frac{1}{2}(AB + AC - BC)$ and $AP = AD + 2AM = AD + (BD + DC - BC) = AB + AC - BC$. This implies that L is the midpoint of AP and the desired result follows. $\square$

The next example has multiple elements that are difficult to work with. Here, we follow cues presented in the diagram and obtain useful constructions (introduced points uniting more than one condition) and reduce the problem to feasible ratio calculations.

Example 3. *(ISL 1996 G3) Let O be the circumcenter and H the orthocenter of an acute-angled triangle ABC such that $BC > CA$. Let F be the foot of the altitude CH of triangle ABC. The perpendicular to the line OF at the point F intersects the line AC at P. Prove that $\angle FHP = \angle BAC$.*

Solution. If the problem statement is true, then $\angle CHP = 180^\circ - \angle BAC$. Based on this angle relationship, intersecting HP with AB creates a cyclic quadrilateral. We reformulate the problem by defining P as the point on AC satisfying $\angle FHP = \angle BAC$ introduce this intersection point and call it D. Our goal is now to show $\angle PFO = 90^\circ$ and the two definitions are therefore equivalent. Since $CHAD$ is cyclic, we have that $\angle CDA = 180^\circ - \angle CHA = \angle CBA$. Since the line OF is difficult to deal with and angles around it have no simple formula, we try to reduce the problem to a condition relating something more directly related to P than OF. We have now that DCB is isosceles and F is the midpoint of BD. If M is the midpoint of AB, then we now note that there is a homothety sending MF to AD with center B and ratio 2. Let E be the image of O under this homothety. Note that $AE = 2OM = CH$. It now suffices to show that $\angle EDA = 90^\circ - \angle PFH$. We now try to reduce this angle condition to length conditions which will be easier to deal with since many angles in the diagram cannot be expressed simply. If G is the intersection of FP with the line through C perpendicular to CH. Since $\angle GCF = \angle EAD = 90^\circ$, it suffices to show that GCF and EAD are similar, which is equivalent to showing that

$$\frac{CH}{AD} = \frac{EA}{AD} = \frac{GC}{CF} = \frac{CP}{PA} \cdot \frac{AF}{CF}$$

Now we resort to a ratio identity for cyclic quadrilaterals. The ratio CP/PA is the ratio of the areas of triangles DCH and DAH. Therefore since $CHAD$ is cyclic, we have that

$$\frac{CP}{PA} = \frac{\sin \angle DCH \cdot CD \cdot CH}{\sin \angle DAH \cdot AD \cdot AH} = \frac{CB \cdot CH}{AD \cdot AH}$$

Therefore the desired result reduces to proving that $AH/AF = BC/CF$ which follows from the fact that AHF and CBF are similar. This completes the proof. $\square$

5 Completing Transformations

One of the most useful techniques in synthetic geometry problems is to recognize a transformation present in a diagram, and introduce whatever points are needed to complete the set of images of points under this transformation. Often this heuristic yields the "magic point" that leads to a quick concise solution. For example, a diagram may contain a parallelogram $ABCD$ in which cases there is a translation mapping AB to DC. A diagram may contain a trapezoid $ABCD$ with $AB \| CD$ in which case there is a homothety mapping AB to CD. The transformations that most commonly appear are spiral similarities, rotations, homotheties and translations. The first few examples illustrate different ways to apply this heuristic for spiral similarities and rotations.

The second example is one direction of Ptolemy's Theorem.

Here we construct similar triangles by applying a spiral similarity with center A mapping the C to D. We let the point B be mapped to P under this map, completing the transformation.

In this example we consider the spiral similarity with center B mapping line CX to the perpendicular bisector of AB in order to obtain the angle we want Y to have at the image Y' of C. We then show that $Y = Y'$.

The next problem illustrates an often useful transformation when there is a midpoint of the side of a triangle. It is often useful to perform a 180° rotation about the midpoint to produce a parallelogram as in the example below which is from Challenging Problems in Geometry.

The next example completes another translation in the same vain as above.

This last example completes a homothety.

Example 4. *(ISL 2006 G2) Let $ABCD$ be a trapezoid with parallel sides $AB > CD$. Points K and L lie on the line segments AB and CD, respectively, so that $\frac{AK}{KB} = \frac{DL}{LC}$. Suppose that there are* *points P and Q on the line segment KL satisfying $\angle APB = \angle BCD$ and $\angle CQD = \angle ABC$. Prove that the points P, Q, B and C are concyclic.*

Solution. Since $ABCD$ is a trapezoid, there is a homothety sending AB to CD as well as one sending AB to DC. We note that the homothety sending AB to DC also sends K to L. Now we complete this homothety in the diagram. Let DA and CB intersect at T and let the homothety with center T bring P to P'. We have that K, P, Q, L and P' are collinear and $PB \| P'C$. Since $\angle DQC + \angle APB = \angle DQC + \angle DP'C = 180°$, we have $DQCP'$ is cyclic. Therefore $\angle QPB = \angle QP'C = \angle QDC = 180° - \angle DQC - \angle QCD = \angle QCB$. The conclusion follows. □

6 Redefining Points

In this section, we build on an idea hinted at in Example 13. Sometimes a point may be defined in a deliberately difficult way in a problem statement. This also was the case in Example 3. Often the key to the solution is to find the "useful way" to define the point and prove that this is in fact the same point. Specifically, if P is a point in the diagram that is difficult to deal with, it is often best to define P' in some other way using a property we think is true of P and then prove that $P' = P$. One thing to note is that this method requires that we have a property of P in mind. Finding out what is true of P is usually the most difficult part of problems that can be solved using this method. Sometimes working backwards is enough, but oftentimes some guesswork, intuition and wishful thinking is necessary.

Often the best conjectures are simple, such as P lies on a line in the diagram, P lies on a circle in the diagram or is concyclic with other points in the diagram, that two lines are parallel or perpendicular, or that two triangles are similar or congruent. It can be useful sometimes to try to eyeball some of these from a well-drawn diagram. Here are is an example.

Example 5. *(ISL 2006 G4) An acute-angled triangle ABC is inscribed in a circle ω. A point P is chosen inside the triangle. Line AP intersects ω at the point A_1. Line BP intersects ω at the point B_1. A line ℓ is drawn through P and intersects BC and AC at the points A_2 and B_2. Prove that the circumcircles of triangles A_1A_2C and B_1B_2C intersect again on line ℓ.*

We want to analyze the second intersection of the circumcircles of triangles A_1A_2C and B_1B_2C. How much we can prove about this intersection Q varies greatly with how we define Q. First let's try defining Q directly as the intersection of the circumcircles of triangles A_1A_2C and B_1B_2C. From this, we know that $\angle CQB_2 = 180° - \angle CB_1B_2$ and $\angle CQA_2 = 180° - \angle CA_1A_2$. What we want is to show that $\angle CQB_2 + \angle CQA_2 = 180°$ which now is equivalent to $\angle CB_1B_2 + \angle CA_1A_2 = 180°$. However, this is not immediately true given the conditions in the problem. This doesn't seem to work. Let's try a different way of defining Q.

Solution. Define Q' as the intersection of the circumcircle of B_1PA_1 and ℓ. From cyclic quadrilaterals, we have

$$\angle B_1Q'P = \angle B_1A_1P = \angle B_1CB_2$$

which implies that Q' is on the circumcircle of B_1B_2C. By a similar argument, we have that Q' is on the circumcircle of A_1A_2C. Together these imply that $Q = Q'$. Thus Q lies on ℓ. □

A solution can also be obtained by defining Q' as the intersection of the circumcircle of B_1B_2C and ℓ. The way we define Q' above can be motivated as follows. We want to define Q' in some way and then use this way to show it lies on circles. The cleanest way to do this is to show the angle conditions for a cyclic quadrilateral. In order to get these angle conditions, one promising approach is to define Q' as the intersection of a circle with something, which in this case is ℓ.

These next examples illustrate this same method applied in more situations. Particularly in Example 19, it is hard to find a clean solution without the observations used to define P'.

Example 6. *(China 2012) In the triangle ABC, $\angle A$ is biggest. On the circumcircle of ABC, let D be the midpoint of arc ABC and E be the midpoint of arc ACB. The circle c_1 passes through A, B and is tangent to AC at A, the circle c_2 passes through A, E and is tangent AD at A. Circles c_1 and c_2 intersect at A and P. Prove that AP bisects $\angle BAC$.*

If the result is true, then by the tangency conditions $\angle APB = 180^\circ - \angle BAC$ and $\angle PBA = 180^\circ - \angle APB - \angle PAB = \frac{1}{2}\angle BAC = \angle PAB$. Therefore if the problem is true, then P lies on the perpendicular bisector of AB. This gives us the hint to try defining P based on this. The method below defines P' as the intersection of c_1 and the perpendicular bisector of AB.

Solution. Let the center of c_1 be O_1 and let the center of c_2 be O_2. Since c_1 is tangent to AC, it follows that $\angle BO_1A = 2\angle BAC$. Since O_1 and E both lie on the perpendicular bisector of AB, it follows that O_1E bisects angle $\angle BO_1A$ which implies that $\angle BO_1A = \angle BAC$ and hence that $\angle BP'E = 90^\circ + \frac{1}{2}\angle BAC$. However, since P' lies on the perpencular bisector EO_1 of AB, A is the reflection of B about EO_1 and $\angle AP'E = \angle BP'E = 90^\circ + \angle BAC$. Since c_2 is tangent to AD and passes through E, it follows that $\angle AO_2E = 2\angle DAE = 180^\circ - \angle BAC$. Combining this with the angle relation above yields that P' lies on c_2. Hence P' lies on both c_1 and c_2 and $P = P'$. Therefore $\angle BAP = \frac{1}{2}\angle BO_1P = \frac{1}{2}\angle BAC$ which implies the result. □

The next example really illustrates the power of redefining a point that is difficult to work with. Here, a relatively simple restatement reduces the problem to simple angle chasing.

Example 7. *(ISL 2002 G3) The circle S has centre O, and BC is a diameter of S. Let A be a point of S such that $\angle AOB < 120^\circ$. Let D be the midpoint of the arc AB which does not contain C. The line through O parallel to DA meets the line AC at I. The perpendicular bisector of OA meets S at E and at F. Prove that I is the incentre of the triangle CEF.*

Solution. We first make several preliminary observations. Since EF is the perpendicular bisector of OA, we have that $AE = OE = OA$ and therefore AOE is equilateral. Similarly, we have that AOF is equilateral which implies that $\angle EOF = 120^\circ$ and $\angle ECF = 60^\circ$. These results also imply that A is the midpoint of arc $\widehat{EF}$ and CA bisects $\angle ECF$. After these preliminary observations, it becomes difficult to work with the point I as defined. The key here is to redefine I to be easier to work with. We now define I' to be the incenter of CEF with the goal of showing that $\angle DAO = \angle AOI'$ since this would imply that $OI' \| AD$ and therefore $I = I'$. At this point, the task becomes far more feasible than before and reduces to angle chasing. First we note that $\angle EOF = 120^\circ$ and $\angle EI'F = 90^\circ + \angle ECF/2 = 120^\circ$ which implies that $EI'OF$ is cyclic. Now we carry out our angle chasing methodically, attempting to eliminate points from consideration as we go. Note that $\angle DAO = 90^\circ - \angle AOD/2 = 90^\circ - \angle ACB/2 = 45^\circ + \angle ABC/2 = 45^\circ + \angle AFC/2$, which is enough to eliminate D and B. Now note that $\angle AOI' = \angle AOE + \angle EOI' = 60^\circ + \angle EFI = 60^\circ + \angle EFC/2$. Since $\angle AFC - \angle EFC = 30^\circ$, we have that $\angle DAO = \angle AOI'$, as desired. □

A remarkably powerful way of redefining points is to try to identify them as *the intersection of a line or circle with another circle*. This yields angle information that often leads to quick solutions. To illustrate this, we outline the solution to what is possibly the hardest geometry problem on the IMO in recent memory.

Example 8. *(IMO 2011) Let ABC be an acute triangle with circumcircle Γ. Let ℓ be a tangent line to Γ, and let ℓ_a, ℓ_b and ℓ_c be the lines obtained by reflecting ℓ in the lines BC, CA and AB, respectively. Show that the circumcircle of the triangle determined by the lines ℓ_a, ℓ_b and ℓ_c is tangent to the circle Γ.*

Exhausting the diagram yields almost nothing promising. The main issue is that we know almost nothing about the point of tangency. The key to the simplest solution to this problem is to find a way to define this supposed point of tangency. We try intersecting circumcircles in order to obtain angle information to prove that the point of intersection lies on Γ, the circumcircle of the triangle determined by the three lines and prove that the circles are tangent at this point.

Solution. Let A', B' and C' be the intersections of ℓ_b and ℓ_c, ℓ_a and ℓ_c, and ℓ_a and ℓ_b, respectively. Let P be the point of tangency between Γ and ℓ and let Q be the reflection of P through BC. Now let T be the second intersection of the circumcircles of $BB'Q$ and $CC'Q$. It can be shown that T lies on Γ and the circumcircle of $A'B'C'$ by angle chasing. Similarly, T can be shown to be a point of tangency between the circles by angle chasing. The angle chasing is made easier by first showing that AA', BB' and CC' meet at the incenter I of $A'B'C'$. $\square$

Example 9. *(2005 G5) Let $\triangle ABC$ be an acute-angled triangle with $AB \neq AC$. Let H be the orthocenter of triangle ABC, and let M be the midpoint of the side BC. Let D be a point on the side AB and E a point on the side AC such that $AE = AD$ and the points D, H, E are on the same line. Prove that the line HM is perpendicular to the common chord of the circumscribed circles of triangle $\triangle ABC$ and triangle $\triangle ADE$.*

Solution. It is a known fact that the line HM passes through P, the point diametrically opposite to A on the circumcircle Γ of ABC. Based on this, it would be convenient if HM passed through the second intersection Q of Γ and the circumcircle of ADE. If this were true, then AQ and the line $\overline{PMHQ}$ would be perpendicular since AP is a diameter of the circumcircle of ABC. At this point, it is not a bad idea to draw one or two precise diagrams and see if our claim is supported. We find that it is and decide to focus on this claim. Proving the claim directly does not seem easy since it is hard to work with the second intersection point while actually using the fact that it lies on both circles. We look for a conjecture easier to prove that arises from our claim. If the claim is true, then $\overline{PMHQ}$ must also pass through the point R diametrically opposite to A on the circumcircle of ADE. Proving this seems more feasible, since it does not involve the second intersection and we work with it first. Treating this new claim as its own subproblem yields the following solution.

Let U and V be the feet of the perpendiculars from B and C to AC and AB. Angle chasing yields that the line $\overline{DHE}$ is the internal bisector of the angle formed by lines BU and CV. It also holds that triangles UHC and VHB are similar. Therefore $UD/DB = VE/EC = t$. If the perpendicular to AB at D intersects HP at R_1, then since $UHPB$ is a trapezoid it follows that $HR_1/R_1P = t$. Similarly if the perpendicular to AC at E intersects HP at R_2, then $HR_2/R_2P = t$ and $R_1 = R_2 = R$. This proves the claim.

Now to complete the solution, take the projection Q' of A onto line $\overline{PMHR}$. Since AR and AP are diameters of the circumcircle of ADE and Γ, it follows that Q' lies on both circles and thus $Q' = Q$. Now it follows that the line $\overline{PMHR}$ is perpendicular to AQ, as desired. $\square$

(2006 G3) Consider a convex pentagon $ABCDE$ such that

$$\angle BAC = \angle CAD = \angle DAE \quad , \quad \angle ABC = \angle ACD = \angle ADE$$

Let P be the point of intersection of the lines BD and CE. Prove that the line AP passes through the midpoint of the side CD.

(2009 G2) Let ABC be a triangle with circumcentre O. The points P and Q are interior points of the sides CA and AB respectively. Let K, L and M be the midpoints of the segments BP, CQ and PQ. respectively, and let Γ be the circle passing through K, L and M. Suppose that the line PQ is tangent to the circle Γ. Prove that $OP = OQ$.

(2000 G3) Let O be the circumcenter and H the orthocenter of an acute triangle ABC. Show that there exist points D, E, and F on sides BC, CA, and AB respectively such that

$$OD + DH = OE + EH = OF + FH$$

and the lines AD, BE, and CF are concurrent.

(2003 G5) Let ABC be an isosceles triangle with $AC = BC$, whose incentre is I. Let P be a point on the circumcircle of the triangle AIB lying inside the triangle ABC. The lines through P parallel to CA and CB meet AB at D and E, respectively. The line through P parallel to AB meets CA and CB at F and G, respectively. Prove that the lines DF and EG intersect on the circumcircle of the triangle ABC.

(2005 G4) Let $ABCD$ be a fixed convex quadrilateral with $BC = DA$ and BC not parallel with DA. Let two variable points E and F lie of the sides BC and DA, respectively and satisfy $BE = DF$. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R. Prove that the circumcircles of the triangles PQR, as E and F vary, have a common point other than P.

(2008 G7) Let $ABCD$ be a convex quadrilateral with BA different from BC. Denote the incircles of triangles ABC and ADC by k_1 and k_2 respectively. Suppose that there exists a circle k tangent to ray BA beyond A and to the ray BC beyond C, which is also tangent to AD and CD. Prove that the common external tangents to k_1 and k_2 intersect on k.

7 Know the Classical Configurations

There are a lot of classical geometry configurations and miscellaneous facts that can help in math contests. Here is a selection of a few that seem to come up over and over again. Many more are included in my other handout. Some of these are difficult and worthwhile to prove on your own.

1. Given a triangle ABC, the intersections of the internal and external bisectors of $\angle BAC$ with the perpendicular bisector of ABC lie on the circumcircle of ABC.

2. Facts related to the orthocenter H of a triangle ABC with circumcircle Γ and center O:

   (a) If D is the point diametrically opposite to A on Γ and M is the midpoint of BC, then M is also the midpoint of HD.

(b) If AH, BH and CH intersect Γ again at D, E and F, then there is a homothety centered at H sending the triangle formed by projecting H onto the sides of ABC to DEF with ratio 2.

(c) If D and E are the intersections of AH with BC and Γ, respectively, then D is the midpoint of HE.

(d) If M is the midpoint of BC then $AH = 2 \cdot OM$.

(e) If BH and CH intersect AC and AB at D and E, and M is the midpoint of BC, then M is the center of the circle through B, D, E and C, and MD and ME are tangent to the circumcircle of ADE.

3. Facts related to the incenter I and excenters I_a, I_b, I_c of ABC with circumcircle Γ:

   (a) If AI intersects Γ at D then $DB = DI = DC$, D is the midpoint of II_a, and II_a is a diameter of the circle with center D which passes through B and C.

   (b) If BI and CI intersect Γ again at D and E, then I is the reflection of A in line DE and if M is the intersection of the external bisector of $\angle BAC$ with Γ, then $DMEI$ is a parallelogram.

   (c) If the incircle and A-excircle of ABC are tangent to BC at D and E, $BD = CE$.

   (d) If M is the midpoint of arc BAC of Γ, then M is the midpoint of I_bI_c and the center of the circle through I_b, I_c, B and C.

4. (Symmedian) Given a triangle ABC such that M is the midpoint of BC, the symmedian from A is the line that is the reflection of AM in the bisector of angle $\angle BAC$.

   (a) If the tangents to the circumcircle Γ of ABC at B and C intersect at N, then N lies on the symmedian from A and $\angle BAM = \angle CAN$.

   (b) If the symmedian from A intersects Γ at D, then $AB/BD = AC/CD$.

5. (Apollonius Circle) Let ABC be a given triangle and let P be a point such that $AB/BC = AP/PC$. If the internal and external bisectors of angle $\angle ABC$ meet line AC at Q and R, then P lies on the circle with diameter QR.

6. (Nine-Point Circle) Given a triangle ABC, let Γ denote the circle passing through the midpoints of the sides of ABC. If H is the orthocenter of ABC, then Γ passes through the midpoints of AH, BH and CH and the projections of H onto the sides of ABC.

7. (Feuerbach's Theorem) The nine-point circle is tangent to the incircle and excircles.

8. (Euler Line) If O, H and G are the circumcenter, orthocenter and centroid of a triangle ABC, then G lies on segment OH with $HG = 2 \cdot OG$.

9. (Euler's Formula) Let O, I and I_a be the circumcenter, incenter and A-excenter of a triangle ABC with circumradius R, inradius r and A-exradius r_a. Then:

   (a) $OI = \sqrt{R(R - 2r)}$.

   (b) $OI_a = \sqrt{R(R + 2r_a)}$.

10. Let ABC be a given triangle with incircle ω and A-excircle ω_a. If ω and ω_a are tangent to BC at M and N, then AN passes through the point diametrically opposite to M on ω and AM passes through the point diametrically opposite to N on ω_a.

11. Let ABC be a triangle with incircle ω which is tangent to BC, AC and AB at D, E and F. Let M be the midpoint of BC. The perpendicular to BC at D, the median AM and the line EF are concurrent.

12. Let ABC be a triangle with incenter I and incircle ω which is tangent to BC, AC and AB at D, E and F. The angle bisector CI intersects FE at a point T on the line adjoining the midpoints of AB and BC. It also holds that $BFTID$ is cyclic and $\angle BTC = 90°$.

13. Let ABC be a triangle with incircle ω and let D and E be the points at which ω is tangent to BC and the A-excircle is tangent to BC. Then AE passes through the point diametrically opposite to D on ω.

14. Let ABC be a triangle with A-excenter I_A and altitutde AD. Let M be the midpoint of AD and let K be the point of tangency between the incircle of ABC and BC. Then I_A, K and M are collinear.

15. Let $ABCD$ be a convex quadrilateral. The four interior angle bisectors of $ABCD$ are concurrent and there exists a circle Γ tangent to the four sides of $ABCD$ if and only if $AB + CD = AD + BC$.

16. (Simson Line) Let M, N and P be the projections of a point Q onto the sides of a triangle ABC. Then Q lies on the circumcircle of ABC if and only if M, N and P are collinear. If Q lies on the circumcircle of ABC, then the reflections of Q in the sides of ABC are collinear and pass through the orthocenter of the triangle.

17. (Butterfly Theorem) Let M be the midpoint of a chord XY of a circle Γ. The chords AB and CD pass through M. If AD and BC intersect chord XY at P and Q, then M is also the midpoint of PQ.

18. (Mixtilinear Incircles) Let ABC be a triangle with circumcircle Γ and let ω be a circle tangent internally to Γ and to AB anc AC at X and Y. Then the incenter of ABC is the midpoint of segment XY.

19. (Curvilinear Incircles) Let ABC be a triangle with circumcircle Γ and let D be a point on segment BC. Let ω be a circle tangent to Γ, DA and DC. If ω is tangent to DA and DC at F and E, then the incenter of ABC lies on FE.

20. (Pole-Polar) Let X lie on the line joining the points of tangency of the tangents from Y to a circle Ω. Then Y lies on the line joining the points of tangency of the tangents from X to Ω.

21. Pascal's, Desargues', Pappus', Feuerbach, other interesting facts to prove, etc? `https://artofproblemsolvi`

8 Suggested Problems

1.

9 More Problems

1. (1998 G1) A convex quadrilateral $ABCD$ has perpendicular diagonals. The perpendicular bisectors of the sides AB and CD meet at a unique point P inside $ABCD$. Prove that the quadrilateral $ABCD$ is cyclic if and only if triangles ABP and CDP have equal areas.

2. (2001 G1) Let A_1 be the center of the square inscribed in acute triangle ABC with two vertices of the square on side BC. Thus one of the two remaining vertices of the square is on side AB and the other is on AC. Points B_1, C_1 are defined in a similar way for inscribed squares with two vertices on sides AC and AB, respectively. Prove that lines AA_1, BB_1, CC_1 are concurrent.

3. (2003 G2) Given three fixed pairwisely distinct points A, B, C lying on one straight line in this order. Let G be a circle passing through A and C whose center does not lie on the line AC. The tangents to G at A and C intersect each other at a point P. The segment PB meets the circle G at Q. Show that the point of intersection of the angle bisector of the angle AQC with the line AC does not depend on the choice of the circle G.

4. (2008 G1) Let H be the orthocenter of an acute-angled triangle ABC. The circle Γ_A centered at the midpoint of BC and passing through H intersects the sideline BC at points A_1 and A_2. Similarly, define the points B_1, B_2, C_1 and C_2. Prove that the six points A_1, A_2, B_1, B_2, C_1 and C_2 are concyclic.

5. (2005 G2) Six points are chosen on the sides of an equilateral triangle ABC: A_1, A_2 on BC, B_1, B_2 on CA and C_1, C_2 on AB, such that they are the vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths. Prove that the lines A_1B_2, B_1C_2 and C_1A_2 are concurrent.

6. (2012 G3) In an acute triangle ABC the points D, E and F are the feet of the altitudes through A, B and C respectively. The incenters of the triangles AEF and BDF are I_1 and I_2 respectively; the circumcenters of the triangles ACI_1 and BCI_2 are O_1 and O_2 respectively. Prove that I_1I_2 and O_1O_2 are parallel.

7. (2007 G3) The diagonals of a trapezoid $ABCD$ intersect at point P. Point Q lies between the parallel lines BC and AD such that $\angle AQD = \angle CQB$, and line CD separates points P and Q. Prove that $\angle BQP = \angle DAQ$.

8. (2009 G4) Given a cyclic quadrilateral $ABCD$, let the diagonals AC and BD meet at E and the lines AD and BC meet at F. The midpoints of AB and CD are G and H, respectively. Show that EF is tangent at E to the circle through the points E, G and H.

9. (2009 G3) Let ABC be a triangle. The incircle of ABC touches the sides AB and AC at the points Z and Y, respectively. Let G be the point where the lines BY and CZ meet, and let R and S be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelogram. Prove that $GR = GS$.

10. (1995 G8) Suppose that $ABCD$ is a cyclic quadrilateral. Let $E = AC \cap BD$ and $F = AB \cap CD$. Denote by H_1 and H_2 the orthocenters of triangles EAD and EBC, respectively. Prove that the points F, H_1, H_2 are collinear.

11. (2007 G4) Consider five points A, B, C, D and E such that $ABCD$ is a parallelogram and $BCED$ is a cyclic quadrilateral. Let ℓ be a line passing through A. Suppose that ℓ intersects the interior of the segment DC at F and intersects line BC at G. Suppose also that $EF = EG = EC$. Prove that ℓ is the bisector of angle DAB.

12. (2011 G4) Let ABC be an acute triangle with circumcircle Ω. Let B_0 be the midpoint of AC and let C_0 be the midpoint of AB. Let D be the foot of the altitude from A and let G be the centroid of the triangle ABC. Let ω be a circle through B_0 and C_0 that is tangent to the circle Ω at a point $X \neq A$. Prove that the points D, G and X are collinear.

13. (2010 G5) Let $ABCDE$ be a convex pentagon such that $BC \parallel AE$, $AB = BC + AE$, and $\angle ABC = \angle CDE$. Let M be the midpoint of CE, and let O be the circumcenter of triangle BCD. Given that $\angle DMO = 90°$, prove that $2\angle BDA = \angle CDE$.

14. (1998 G5) Let ABC be a triangle, H its orthocenter, O its circumcenter, and R its circumradius. Let D be the reflection of the point A across the line BC, let E be the reflection of the point B across the line CA, and let F be the reflection of the point C across the line AB. Prove that the points D, E and F are collinear if and only if $OH = 2R$.

15. (1999 G6) Two circles Ω_1 and Ω_2 touch internally the circle Ω in M and N and the center of Ω_2 is on Ω_1. The common chord of the circles Ω_1 and Ω_2 intersects Ω in A and B. MA and MB intersects Ω_1 in C and D. Prove that Ω_2 is tangent to CD.

16. (2005 G6) Let ABC be a triangle, and M the midpoint of its side BC. Let γ be the incircle of triangle ABC. The median AM of triangle ABC intersects the incircle γ at two points K and L. Let the lines passing through K and L, parallel to BC, intersect the incircle γ again in two points X and Y. Let the lines AX and AY intersect BC again at the points P and Q. Prove that $BP = CQ$.

17. (2009 G6) Let the sides AD and BC of the quadrilateral $ABCD$ (such that AB is not parallel to CD) intersect at point P. Points O_1 and O_2 are circumcenters and points H_1 and H_2 are orthocenters of triangles ABP and CDP, respectively. Denote the midpoints of segments O_1H_1 and O_2H_2 by E_1 and E_2, respectively. Prove that the perpendicular from E_1 on CD, the perpendicular from E_2 on AB and the lines H_1H_2 are concurrent.

18. (2011 G3) Let $ABCD$ be a convex quadrilateral whose sides AD and BC are not parallel. Suppose that the circles with diameters AB and CD meet at points E and F inside the quadrilateral. Let ω_E be the circle through the feet of the perpendiculars from E to the lines AB, BC and CD. Let ω_F be the circle through the feet of the perpendiculars from F to the lines CD, DA and AB. Prove that the midpoint of the segment EF lies on the line through the two intersections of ω_E and ω_F.

19. (2006 G9) Points A_1, B_1, C_1 are chosen on the sides BC, CA, AB of a triangle ABC respectively. The circumcircles of triangles AB_1C_1, BC_1A_1, CA_1B_1 intersect the circumcircle of triangle ABC again at points A_2, B_2, C_2 respectively ($A_2 \neq A, B_2 \neq B, C_2 \neq C$). Points A_3, B_3, C_3 are symmetric to A_1, B_1, C_1 with respect to the midpoints of the sides BC, CA, AB respectively. Prove that the triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.

20. (2012 G6) Let ABC be a triangle with circumcenter O and incenter I. The points D, E and F on the sides BC, CA and AB respectively are such that $BD + BF = CA$ and $CD + CE = AB$. The circumcircles of the triangles BFD and CDE intersect at $P \neq D$. Prove that $OP = OI$.

21. (2007 G8) Point P lies on side AB of a convex quadrilateral $ABCD$. Let ω be the incircle of triangle CPD, and let I be its incenter. Suppose that ω is tangent to the incircles of triangles APD and BPC at points K and L, respectively. Let lines AC and BD meet at E, and let lines AK and BL meet at F. Prove that points E, I, and F are collinear.

22. (2009 G8) Let $ABCD$ be a circumscribed quadrilateral. Let g be a line through A which meets the segment BC in M and the line CD in N. Denote by I_1, I_2 and I_3 the incenters of $\triangle ABM$, $\triangle MNC$ and $\triangle NDA$, respectively. Prove that the orthocenter of $\triangle I_1I_2I_3$ lies on g.

23. (2004 G8) Given a cyclic quadrilateral $ABCD$, let M be the midpoint of the side CD, and let N be a point on the circumcircle of triangle ABM. Assume that the point N is different from the point M and satisfies $\frac{AN}{BN} = \frac{AM}{BM}$. Prove that the points E, F, N are collinear, where $E = AC \cap BD$ and $F = BC \cap DA$.

24. (2011 G6) Let ABC be a triangle with $AB = AC$ and let D be the midpoint of AC. The angle bisector of $\angle BAC$ intersects the circle through D, B and C at the point E inside the triangle ABC. The line BD intersects the circle through A, E and B in two points B and F. The lines AF and BE meet at a point I, and the lines CI and BD meet at a point K. Show that I is the incentre of triangle KAB.

25. (2011 G7) Let $ABCDEF$ be a convex hexagon all of whose sides are tangent to a circle ω with centre O. Suppose that the circumcircle of triangle ACE is concentric with ω. Let J be the foot of the perpendicular from B to CD. Suppose that the perpendicular from B to DF intersects the line EO at a point K. Let L be the foot of the perpendicular from K to DE. Prove that $DJ = DL$.

26. (2012 G2) Let $ABCD$ be a cyclic quadrilateral whose diagonals AC and BD meet at E. The extensions of the sides AD and BC beyond A and B meet at F. Let G be the point such that $ECGD$ is a parallelogram, and let H be the image of E under reflection in AD. Prove that D, H, F, G are concyclic.

27. (2012 G4) Let ABC be a triangle with $AB \neq AC$ and circumcenter O. The bisector of $\angle BAC$ intersects BC at D. Let E be the reflection of D with respect to the midpoint of BC. The lines through D and E perpendicular to BC intersect the lines AO and AD at X and Y respectively. Prove that the quadrilateral $BXCY$ is cyclic.

28. (2013 G3) Let ABC be a triangle with $\angle B > \angle C$. Let P and Q be two different points on line AC such that $\angle PBA = \angle QBA = \angle ACB$ and A is located between P and C. Suppose that there exists an interior point D of segment BQ for which $PD = PB$. Let the ray AD intersect the circle ABC at $R \neq A$. Prove that $QB = QR$.

29. (2013 G5) Let $ABCDEF$ be a convex hexagon with $AB = DE$, $BC = EF$, $CD = FA$, and $\angle A - \angle D = \angle C - \angle F = \angle E - \angle B$. Prove that the diagonals AD, BE, and CF are concurrent.

30. (2014 G3) Let Ω and O be the circumcircle and the circumcentre of an acute-angled triangle ABC with $AB > BC$. The angle bisector of $\angle ABC$ intersects Ω at $M \neq B$. Let Γ be the circle with diameter BM. The angle bisectors of $\angle AOB$ and $\angle BOC$ intersect Γ at points P and Q, respectively. The point R is chosen on the line PQ so that $BR = MR$. Prove that $BR \parallel AC$.

31. (2015 G3) Let ABC be a triangle with $\angle C = 90°$, and let H be the foot of the altitude from C. A point D is chosen inside the triangle CBH so that CH bisects AD. Let P be the intersection point of the lines BD and CH. Let ω be the semicircle with diameter BD that meets the segment CB at an interior point. A line through P is tangent to ω at Q. Prove that the lines CQ and AD meet on ω.

10 Hints

If there is a theorem or fact that is useful, I have tried to indicate it with the label TL.

1.